

SVD

Singular Value
Decomposition

→ Analyzing Behavior
Characterizing the
MIMO channel.

H ← $r \times t$ channel matrix

r rows
 t columns.

$$H = U \Sigma V^H \quad r \geq t$$

$$\underbrace{\begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \dots & \bar{u}_t \end{bmatrix}}_{r \times t} \underbrace{\begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & & \\ \vdots & & \ddots & \\ 0 & & & \sigma_t \end{bmatrix}}_{t \times t} \underbrace{\begin{bmatrix} \bar{v}_1^H \\ \bar{v}_2^H \\ \vdots \\ \bar{v}_t^H \end{bmatrix}}_{t \times t}$$

$$\bar{u}_i^H \bar{u}_j^H = 0 \text{ if } i \neq j$$

$$\|\bar{u}_i\|^2 = 1$$

$$\bar{u}_i^H \bar{u}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

The columns $\bar{u}_i$ are orthonormal.

$$U^H U = \begin{bmatrix} \bar{u}_1^H \\ \bar{u}_2^H \\ \vdots \\ \bar{u}_t^H \end{bmatrix} [\bar{u}_1 \ \bar{u}_2 \ \dots \ \bar{u}_t] = \begin{bmatrix} \cancel{\bar{u}_1^H \bar{u}_1}^1 \cancel{\bar{u}_1^H \bar{u}_2}^0 & \dots \\ \cancel{\bar{u}_2^H \bar{u}_1}^0 \cancel{\bar{u}_2^H \bar{u}_2}^1 & \dots \\ \vdots & \ddots \\ \vdots & \dots & \cancel{\bar{u}_t^H \bar{u}_t}^1 \end{bmatrix} = \begin{bmatrix} 1 & \dots & 0 \\ 0 & 1 & \dots \\ \vdots & \vdots & \ddots & 1 \end{bmatrix} = I_{t \times t}$$

$$U^H U = I_{t \times t}$$

orthonormal columns.

$$V^H V = V V^H = I_{t \times t}$$

V is a Unitary Matrix

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & & \\ \vdots & & \ddots & \\ 0 & & & \sigma_t \end{bmatrix}$$

number of non zero singular values = rank of matrix

$$\sigma_1, \sigma_2, \dots, \sigma_t \geq 0$$

Singular values are non-negative.

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_t$$

Singular values are arranged in decreasing order of magnitude.

Example:

$$H = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}}_{\substack{U \\ 2 \times 1}} \underbrace{[\sqrt{2}]}_{\Sigma_{1 \times 1}} \underbrace{[1]}_{\substack{V \\ 1 \times 1}}$$

$\downarrow$
 $\begin{matrix} 2 \times 1 \\ r=2 \end{matrix} \leftarrow t=1$

$$U^H U = 1$$

$$\sigma_1 = \sqrt{2} > 0$$

$$V = [1] \quad V^H V = V V^H = I_{1 \times 1}$$

Example:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

NOT a
valid SVD.

$$H = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_U \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{5} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{V^H}$$

$$\sigma_1 = 1 \quad \sigma_2 = \sqrt{5} \quad \underline{\sigma_1 < \sigma_2}$$

violates SVD
condition

$$H = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{5} \\ 1 & 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_U \underbrace{\begin{bmatrix} \sqrt{5} & 0 \\ 0 & 1 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_V$$

Valid
SVD.

$$U^H U = I \quad V^H V = V V^H = I$$

$$\sigma_1 = \sqrt{5} \quad \sigma_2 = 1$$

$$\sigma_1 > \sigma_2$$